Example: Profit maximization.

Joint weekly demand functions for a firm's competing products:

$$Q_A = 100 - 3P_A + 2P_B$$
$$Q_B = 60 + 2P_A - 2P_B$$

Weekly cost of producing Q_A units of product A and Q_B units of product B:

$$C = 20Q_A + 30Q_B + 1200$$

Firm's weekly profit function

$$\begin{aligned}
\Pi &= P_A Q_A + P_B Q_B - C \\
&= P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B) \\
&\qquad - (20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200) \\
&= -3P_A^2 + 4P_A P_B - 2P_B^2 + 100P_A + 80P_B - 5000
\end{aligned}$$

Weekly profit function

$$\Pi = -3P_A^2 + 4P_AP_B - 2P_B^2 + 100P_A + 80P_B - 5000$$

First order conditions for max:

$$\Pi_{P_A} = 0 \quad \Longrightarrow \quad -6P_A + 4P_B + 100 = 0$$
$$\Pi_{P_B} = 0 \quad \Longrightarrow \quad 4P_A - 4P_B + 80 = 0$$

Adding the two equations together gives an equation for the critical P_A value:

$$-2P_A + 180 = 0 \implies P_A^* = 90.$$

Substituting this in the second equation ($\Pi_{P_B} = 0$) yields the critical P_B value:

$$4 \cdot 90 - 4P_B + 80 = 0 \implies -4p_B + 440 = 0 \implies P_B^* = 110.$$

The corresponding critical weekly outputs are

$$Q_A^* = 100 - 3P_A^* + 2P_B^* = 50 \quad \text{and} \quad Q_B^* = 60 + 2P_A^* - 2P_B^* = 20.$$

The critical weekly revenue is

$$R^* = P_A^* Q_A^* + P_B^* Q_B^* = 6700,$$

the critical weekly cost is

$$C^* = 20Q_A^* + 30Q_B^* + 1200 = 3800$$

and the critical weekly profit is

$$\Pi^* = R^* - C^* = 2900.$$

The critical question:

*Is Π^* the **maximum** weekly profit?*

Second Derivative Test - Two Variable Case

(✻) ***First order conditions*** for a local extreme value of the function $z = f(x, y)$ at the point (a, b):

$$f_x(a, b) = 0 = f_y(a, b)$$

(✻) If the first-order conditions are satisfied at (a, b), then the quadratic Taylor polynomial for $f(x, y)$ centered at (a, b) looks like this:

$$\begin{aligned} T_2(x, y) &= f(a, b) + \frac{f_{xx}(a, b)}{2}(x-a)^2 + f_{xy}(a, b)(x-a)(y-b) \\ &\qquad + \frac{f_{yy}(a, b)}{2}(y-b)^2 \\ &= f(a, b) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \end{aligned}$$

where $A = \dfrac{f_{xx}(a, b)}{2}$, $B = f_{xy}(a, b)$ and $C = \dfrac{f_{yy}(a, b)}{2}$.

If (x, y) is close to (a, b), then $f(x, y) \approx T_2(x, y)$, and therefore

$$\begin{aligned} f(x, y) - f(a, b) &\approx T_2(x, y) - f(a, b) \\ &= A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2. \end{aligned}$$

There are three cases to consider:

1. $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \geq 0$ for all (x, y).

2. $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \leq 0$ for all (x, y).

3. $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$ takes *positive* values at some points and *negative* values at other points.

Each of these cases characterizes the critical value differently...

1. If $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \geq 0$ for all (x,y).

Then $f(x,y) - f(a,b) \approx A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \geq 0$ for all points (x,y) that are sufficiently close to (a,b), so that $f(a,b)$ is a ***local minimum*** value.

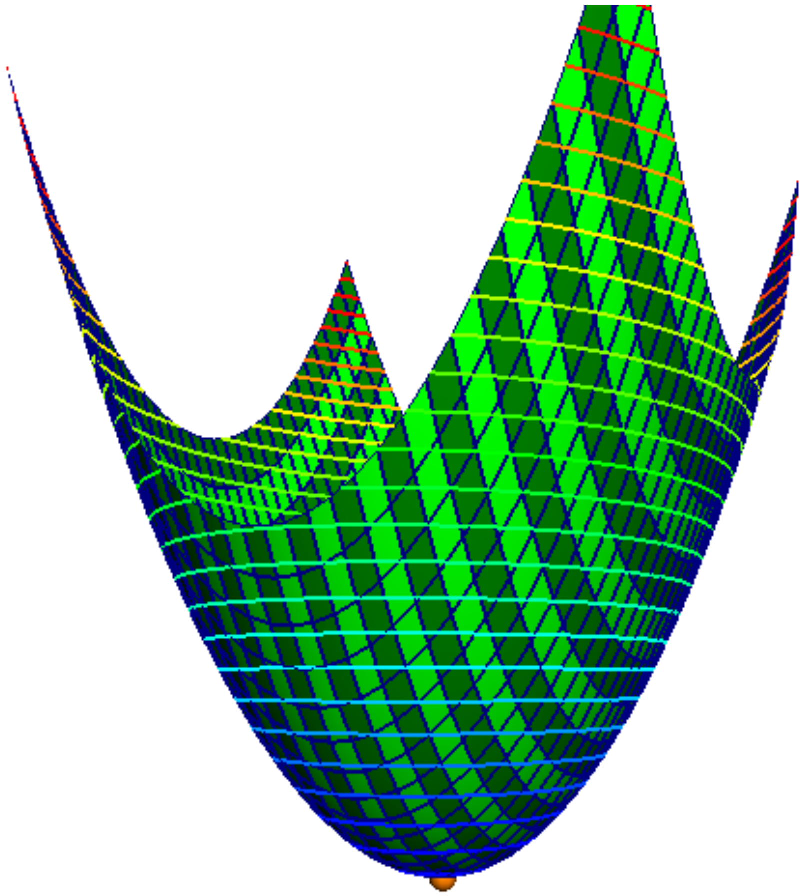
2. If $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \leq 0$ for all (x,y).

Then $f(x,y) - f(a,b) \approx A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \leq 0$ for all points (x,y) that are sufficiently close to (a,b), so that $f(a,b)$ is a ***local maximum*** value.

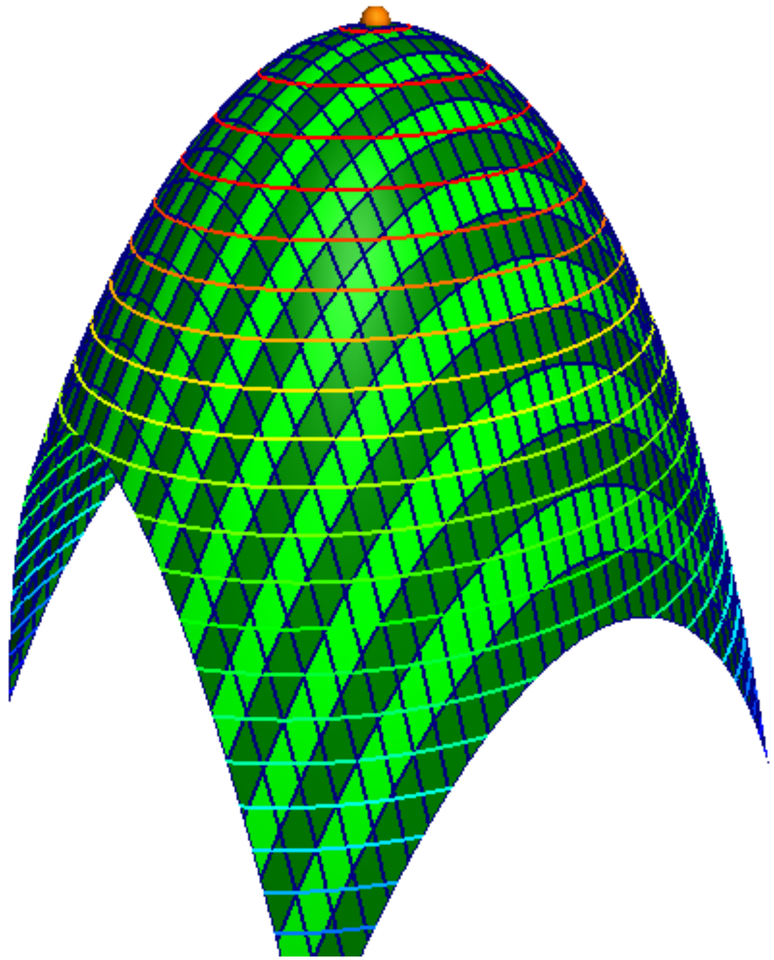
3. If $A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$ takes both positive and negative values,

Then $f(x,y) - f(a,b) \geq 0$ at some points (x,y) that are close to (a,b), and $f(x,y) - f(a,b) \leq 0$ at other points (x,y) that are close to (a,b). In this case, $f(a,b)$ is *neither a local maximum nor a local minimum value* and the point $(a,b,f(a,b))$ is called a ***saddle point*** on the graph $z = f(x,y)$.

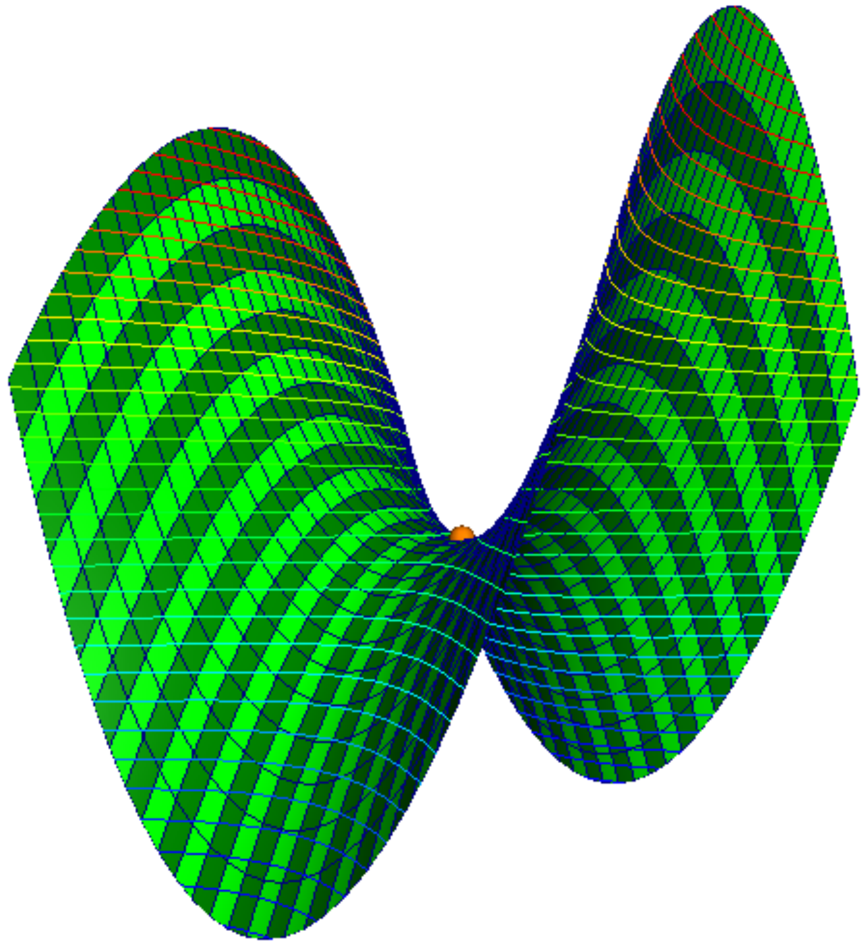
In case 1, in the vicinity of $(a, b, f(a, b))$ (the orange dot), the graph of $z = f(x, y)$ looks like this:



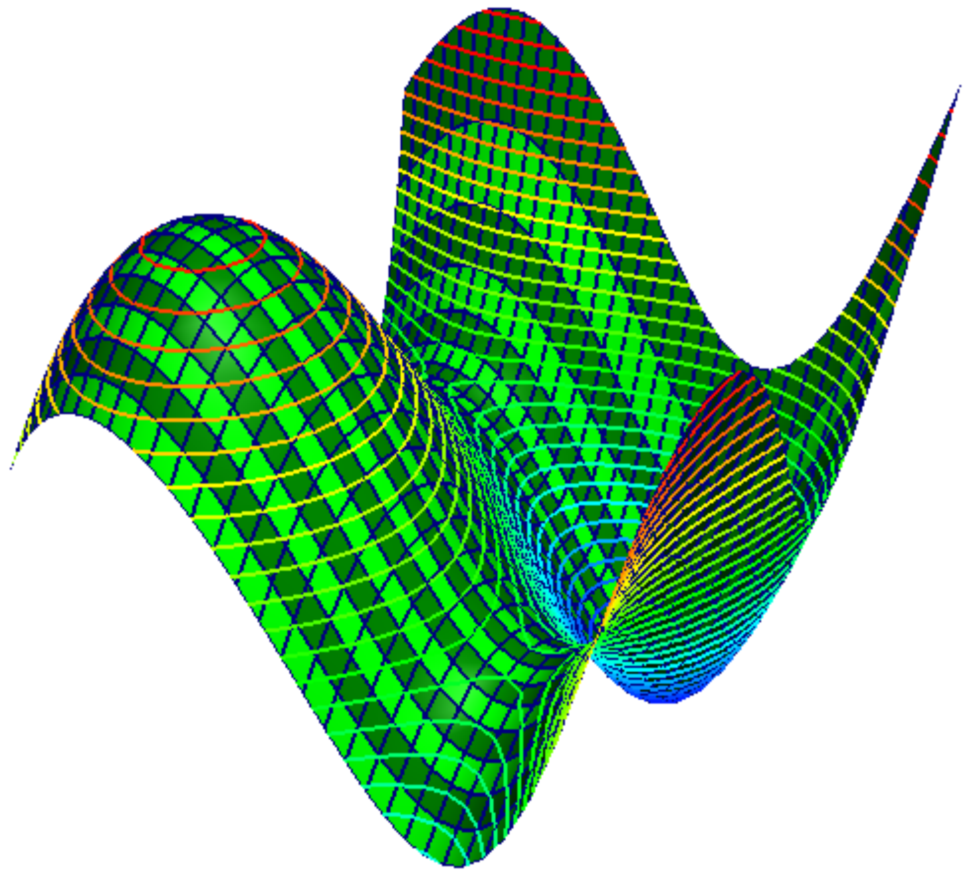
In case 2, in the vicinity of $(a, b, f(a, b))$, the graph of $z = f(x, y)$ looks like this:



In case 3, in the vicinity of $(a, b, f(a, b))$, the graph of $z = f(x, y)$ looks like this:



In general, $z = f(x, y)$ may have multiple critical points and exhibit different behavior at different critical points, as in the case of function $z = 10 - (x-1)^2 - (y-2)^2 + \frac{2}{81}(x-1)^4 + \frac{2}{81}(y-2)^4$, whose graph is depicted below.



Question: *Given a critical point (a, b), how do we determine which case we are in?*

Answer: *We use the algebraic identity:*

$$\begin{aligned} A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \\ &= A\left(\left[(x-a) + \frac{B}{2A}(y-b)\right]^2 + (4AC - B^2)\frac{(y-b)^2}{4A^2}\right) \\ &= A(U^2 + DV^2) \end{aligned}$$

where

$$A = \frac{f_{xx}(a,b)}{2}, \qquad B = f_{xy}(a,b), \qquad C = \frac{f_{yy}(a,b)}{2},$$

$$U = \left[(x-a) + \frac{B}{2A}(y-b)\right], \qquad V = \frac{y-b}{2A}$$

and

$$D = 4AC - B^2 = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2.$$

Conclusion: ***If*** (x, y) ***is close to*** (a, b)***, then***

$$f(x, y) - f(a, b) \approx A\left(U^2 + DV^2\right)$$

where $A = f_{xx}(a, b)$ and $D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$.

Therefore...

1. If $D > 0$ and $A > 0$, then $A\left(U^2 + DV^2\right) \geq 0$, and if (x, y) is close to (a, b) then

$$f(x, y) - f(a, b) \geq 0,$$

so $f(a, b)$ is a local **minimum** value.

2. If $D > 0$ and $A < 0$, then $A\left(U^2 + DV^2\right) \leq 0$, and if (x, y) is close to (a, b) then

$$f(x, y) - f(a, b) \leq 0,$$

so $f(a, b)$ is a local **maximum** value.

Conclusion: ***If*** (x, y) ***is close to*** (a, b)***, then***

$$f(x,y) - f(a,b) \approx A\left(U^2 + DV^2\right)$$

where $A = f_{xx}(a,b)$ and $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$.

Therefore...

3. If $D < 0$, then $\left(U^2 + DV^2\right) > 0$ if $U \neq 0$ and $V = 0$, but $\left(U^2 + DV^2\right) < 0$ if $V \neq 0$ and $U = 0$.

Therefore there are points (x, y) close to (a, b) where

$$f(x,y) - f(a,b) \approx A\left(U^2 + DV^2\right) > 0,$$

and there are also (different) points (x, y) close to (a, b) where

$$f(x,y) - f(a,b) \approx A\left(U^2 + DV^2\right) < 0,$$

This means that, in this case, $f(a, b)$ is neither a maximum nor a minimum value.

The second derivative test for two variables:

If $f_x(a,b) = 0$ *and* $f_y(a,b) = 0$ *(i.e., if* (a,b) *is critical point), then find the second order partial derivatives,* $f_{xx}(a,b), f_{xy}(a,b)$ *and* $f_{yy}(a,b)$ *and the* ***discriminant***

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b),$$

and then analyze:

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local ***minimum*** value.

2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local ***maximum*** value.

3. If $D(a,b) < 0$ then $f(a,b)$ is neither a local minimum value nor a local maximum value — $(a,b,f(a,b))$ is a saddle point on the graph $z = f(x,y)$.

(*) If $D(a,b) = 0$, then the second derivative test yields no conclusions.

Example. Profit maximization (continued).

We found that the critical prices for the profit function

$$\Pi = -3P_A^2 + 4P_A P_B - 2P_B^2 + 100P_A + 80P_B - 5000$$

are $P_A^* = 90$ and $P_B^* = 110$, and the corresponding critical profit is

$$\Pi^* = 2900.$$

We will use the second derivative test to *verify* that the critical profit is indeed a *maximum* value. The first order derivatives are

$$\Pi_{P_A} = -6P_A + 4P_B + 100 \quad \text{and} \quad \Pi_{P_B} = 4P_A - 4P_B + 80$$

so the second order derivatives are

$$\Pi_{P_A P_A} = -6, \quad \Pi_{P_A P_B} = 4 \text{ and } \Pi_{P_B P_B} = -4.$$

The discriminant is

$$D = \Pi_{P_A P_A}\Pi_{P_B P_B} - \Pi_{P_A P_B}^2 = 24 - 16 = 8 > 0$$

and $\Pi_{P_A P_A} < 0$, so Π^* is a maximum, as hoped for.

Example: Find the critical points and the critical values of

$$f(x, y) = x^2 + y^2 - xy + x^3.$$

The partial derivatives are

$$f_x = 2x - y + 3x^2 \quad \text{and} \quad f_y = 2y - x.$$

and solving the pair of equations

$$\begin{aligned} 2x - y + 3x^2 &= 0 \\ 2y - x &= 0 \end{aligned}$$

we find that the critical points are $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (-1/2, -1/4)$, with critical values $f(0, 0) = 0$ and $f(-1/2, -1/4) = 1/16$.

On to the second derivative test:

Discriminant: $f_{xx} = 2 + 6x, \quad f_{xy} = -1 \quad \text{and} \quad f_{yy} = 2$, so

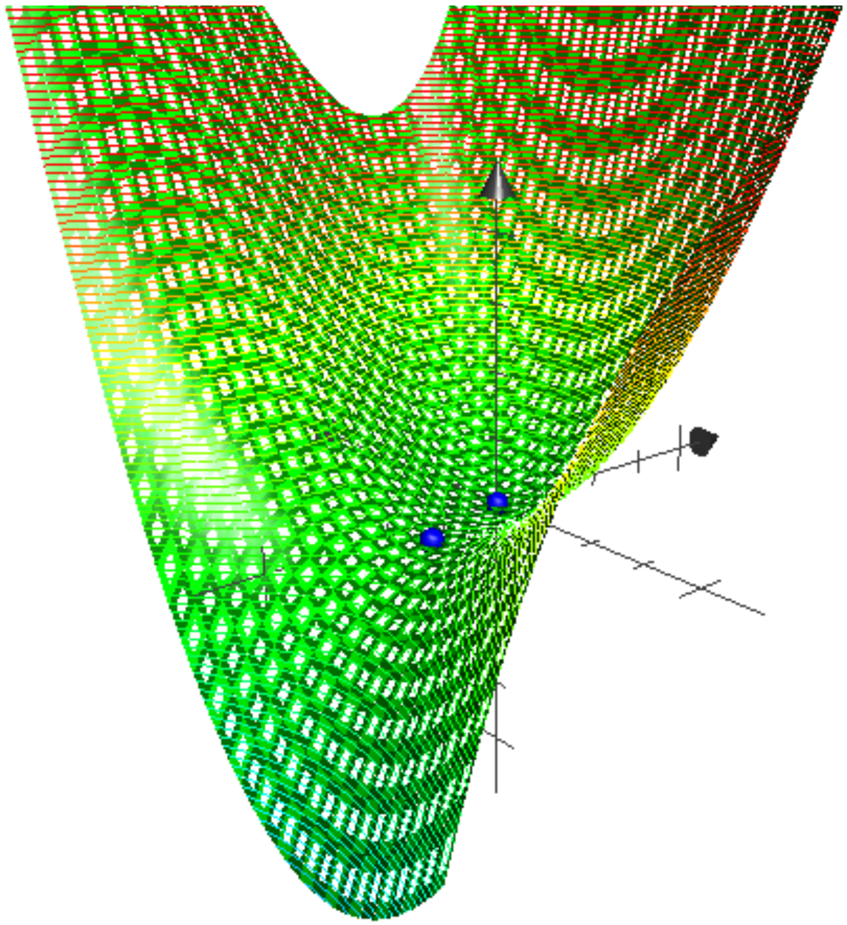
$$D(x, y) = \overbrace{2(2 + 6x)}^{f_{xx} f_{yy}} - \overbrace{(-1)^2}^{f_{xy}^2} = 12x + 3.$$

Analysis:

(*) $D(0, 0) = 3 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $f(0, 0) = 0$ is a *relative minimum value.*

(*) $D(-1/2, -1/4) = -3 < 0$, so $f(-1/2, -1/4) = 5/16$ is *neither a minimum nor a maximum value.*

Graph of $z = x^2 + y^2 - xy + x^3$



(*) The two blue dots are located at the critical points on the graph.