Taylor polynomials in one variable:

The *linear* Taylor polynomial for $y = f(t)$, centered at t_0 is

$$T_1(t) = f(t_0) + f'(t_0) \cdot (t - t_0).$$

It has the properties

- $T_1(t_0) = f(t_0)$ and
- $T_1'(t_0) = f'(t_0)$.

The *quadratic* Taylor polynomial $T_2(t)$ for $f(t)$, centered at t_0 is given by

$$T_2(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(t_0)}{2}(t - t_0)^2,$$

which has the properties

- $T_1(t_0) = f(t_0)$,
- $T_1'(t_0) = f'(t_0)$ and
- $T_1''(t_0) = f''(t_0)$.

These key properties mean that $T_1(t)$ and even more so, $T_2(t)$ behave very much like $f(t)$ at t_0, and as a consequence, the approximations

$$f(t) \approx T_1(t) \quad \text{and} \quad f(t) \approx T_2(t)$$

are very accurate when t is close to t_0, with the quadratic approximation usually being better.

Our goal today is to generalize these ideas to functions of several variables.

We have already introduced the *linear* Taylor polynomial for $f(x, y)$ centered at a point (x_0, y_0),

$$T_1(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which has analogous properties to T_1 in the one-variable case:

- $T_1(x_0, y_0) = f(x_0, y_0)$,
- $T_{1x}(x_0, y_0) = f_x(x_0, y_0)$ and
- $T_{1y}(x_0, y_0) = f_y(x_0, y_0)$

and as a consequence, gives a good approximation $f(x, y) \approx T_1(x, y)$ if (x, y) is close (in the plane) to (x_0, y_0).

To generalize the quadratic Taylor polynomial to functions of two variables, we'll look for a quadratic function in two variables $T_2(x, y)$ that satisfies conditions analogous to those satisfied by $T_2(t)$ in the one-variable case.

Quadratic functions in two variables generally have the form

$$P(x,y) = \overbrace{\alpha}^{\text{constant}} + \overbrace{\beta x + \gamma y}^{\text{linear terms}} + \overbrace{\delta x^2 + \varepsilon xy + \phi y^2}^{\text{quadratic terms}}.$$

For the purposes of finding the quadratic Taylor polynomial $T_2(x,y)$ for the function $f(x,y)$, centered at the point (x_0, y_0), it is more convenient to write it as

$$\begin{aligned} T_2(x,y) = A + &B(x-x_0) + C(y-y_0) \\ &+ D(x-x_0)^2 + E(x-x_0)(y-y_0) + F(y-y_0)^2, \end{aligned}$$

because it makes finding the values of the coefficients $(A, \ldots, F)$ *much* easier. The conditions that we want T_2 to satisfy are:

- $T_2(x_0,y_0) = f(x_0,y_0)$
- $T_{2x}(x_0,y_0) = f_x(x_0,y_0)$
- $T_{2y}(x_0,y_0) = f_y(x_0,y_0)$
- $T_{2xx}(x_0,y_0) = f_{xx}(x_0,y_0)$
- $T_{2xy}(x_0,y_0) = f_{xy}(x_0,y_0)$
- $T_{2yy}(x_0,y_0) = f_{yy}(x_0,y_0)$

These conditions lead to the following equations for the coefficients of $T_2(x,y) = A+B(x-x_0)+C(y-y_0)+D(x-x_0)^2+E(x-x_0)(y-y_0)+F(y-y_0)^2$.

Constant term:

$$f(x_0,y_0) = T_2(x_0,y_0) = A+0+0+0+0+0 = A \implies A = f(x_0,y_0).$$

Linear terms:

$$T_{2x} = B + 2D(x-x_0) + E(y-y_0)$$

and

$$T_{2y} = C + E(x-x_0) + 2F(y-y_0),$$

so

$$f_x(x_0,y_0) = T_{2x}(x_0,y_0) = B+0+0 \implies B = f_x(x_0,y_0)$$

$$f_y(x_0,y_0) = T_{2y}(x_0,y_0) = C+0+0 \implies C = f_y(x_0,y_0)$$

Quadratic terms:

$$T_{2xx} = 2D, \quad T_{2xy} = E \quad \text{and} \quad T_{2yy} = 2F,$$

so

$$f_{xx}(x_0, y_0) = T_{2xx}(x_0, y_0) = 2D \implies D = \frac{f_{xx}(x_0, y_0)}{2}$$

$$f_{xy}(x_0, y_0) = T_{2xy}(x_0, y_0) = E \implies E = f_{xy}(x_0, y_0)$$

$$f_{yy}(x_0, y_0) = T_{2yy}(x_0, y_0) = 2F \implies F = \frac{f_{yy}(x_0, y_0)}{2}$$

Conclusion:

The quadratic Taylor polynomial for $f(x,y)$, centered at (x_0,y_0) is

$$\begin{aligned}T_2(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\\ + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2\\ + f_{xy}(x_0,y_0)(x-x_0)(y-y_0)\\ + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2\end{aligned}$$

Quadratic approximation: If (x,y) is close to (x_0,y_0), then

$$f(x,y) \approx T_2(x,y).$$

Comment: '(x,y) is close to (x_0,y_0)' means that the distance in the plane between these two points is small. If both $|x-x_0|$ is small and $|y-y_0|$ is small, then the distance between (x,y) and (x_0,y_0),

$$\sqrt{(x-x_0)^2+(y-y_0)^2},$$

will also be small.

Example. Find the quadratic Taylor polynomial for

$$f(x,y) = (2x+5y)^{1/2}$$

centered at the point $(x_0, y_0) = (3, 2)$.

This amounts to (a) finding the coefficients and (b) putting the pieces together correctly. To find the coefficients, we need to find the first and second order partial derivatives of $f(x,y)$:

$$f_x = (2x+5y)^{-1/2},\ f_y = \frac{5}{2}(2x+5y)^{-1/2},\ f_{xx} = -(2x+5y)^{-3/2},$$

$$f_{xy} = -\frac{5}{2}(2x+5y)^{-3/2} \text{ and } f_{yy} = -\frac{25}{4}(2x+5y)^{-3/2}.$$

Therefore

$$f(3,2) = 16^{1/2} = 4, \quad f_x(3,2) = 16^{-1/2} = \frac{1}{4}, \quad f_y(3,2) = \frac{5}{2} \cdot 16^{-1/2} = \frac{5}{8}$$

$$f_{xx}(3,2) = -16^{-3/2} = -\frac{1}{64}, \quad f_{xy}(3,2) = -\frac{5}{2} \cdot 16^{-3/2} = -\frac{5}{128}$$

$$\text{and} \quad f_{yy}(3,2) = -\frac{25}{4} \cdot 16^{-3/2} = -\frac{25}{256}.$$

So

$$T_2(x,y) = 4+\frac{1}{4}(x-3)+\frac{5}{8}(y-2)-\frac{1}{128}(x-3)^2-\frac{5}{128}(x-3)(y-2)-\frac{25}{512}(y-2)^2$$

We can use this to approximate

$$\sqrt{17} = f(3.25, 2.1)\ldots$$

$$\begin{aligned} f(3.25, 2.1) &= 4 + \frac{1}{4}\cdot\frac{1}{4} + \frac{5}{8}\cdot\frac{1}{10} - \frac{1}{128}\cdot\frac{1}{16} - \frac{5}{128}\cdot\frac{1}{4}\cdot\frac{1}{10} - \frac{25}{512}\cdot\frac{1}{100} \\ &= 4 + \frac{1}{16} + \frac{1}{16} - \frac{1}{2048} - \frac{1}{1024} - \frac{1}{2048} \\ &= \frac{8192 + 128 + 128 - 1 - 2 - 1}{2048} \\ &= \frac{2111}{512} \end{aligned}$$

Quadratic approximation: $\sqrt{17} \approx \frac{2111}{512} = 4.123046875$

Calculator: $\sqrt{17} = 4.1231056\ldots$

Error: $\left|\sqrt{17} - \frac{2111}{512}\right| < 0.00006.$

Using the same process, we can find quadratic Taylor polynomials for functions of any number of variables.

For example, the quadratic Taylor polynomial for $f(x, y, z)$ centered at (x_0, y_0, z_0) is given by

$$\begin{aligned} T_2(x, y, z) = f(x_0, y_0, z_0) &+ f_x(x_0, y_0, z_0)(x - x_0) \\ &+ f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \\ &+ \frac{f_{xx}(x_0, y_0, z_0)}{2}(x - x_0)^2 \\ &+ \frac{f_{yy}(x_0, y_0, z_0)}{2}(y - y_0)^2 \\ &+ \frac{f_{zz}(x_0, y_0, z_0)}{2}(z - z_0)^2 \\ &+ f_{xy}(x_0, y_0, z_0)(x - x_0)(y - y_0) \\ &+ f_{xz}(x_0, y_0, z_0)(x - x_0)(z - z_0) \\ &+ f_{yz}(x_0, y_0, z_0)(y - y_0)(z - z_0) \end{aligned}$$

Relax..!

(*) We won't be computing specific quadratic Taylor polynomials for approximation (or any other) purposes.

(*) We ***will*** be using the quadratic Taylor polynomial in two variables to understand the *second derivative test* in two variables. To this end, the key observation is the following.

If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (we say that (x_0, y_0) is a *critical point* in this case), then the quadratic Taylor polynomial for $f(x, y)$ centered at (x_0, y_0) is

$$\begin{aligned} T_2(x, y) = f(x_0, y_0) + \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2 \end{aligned}$$

It follows in this case that if (x, y) is close to (x_0, y_0), then

$$\begin{aligned} f(x, y) - f(x_0, y_0) &\approx T_2(x, y) - f(x_0, y_0) \\ &= \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\ &\quad + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2 \end{aligned}$$