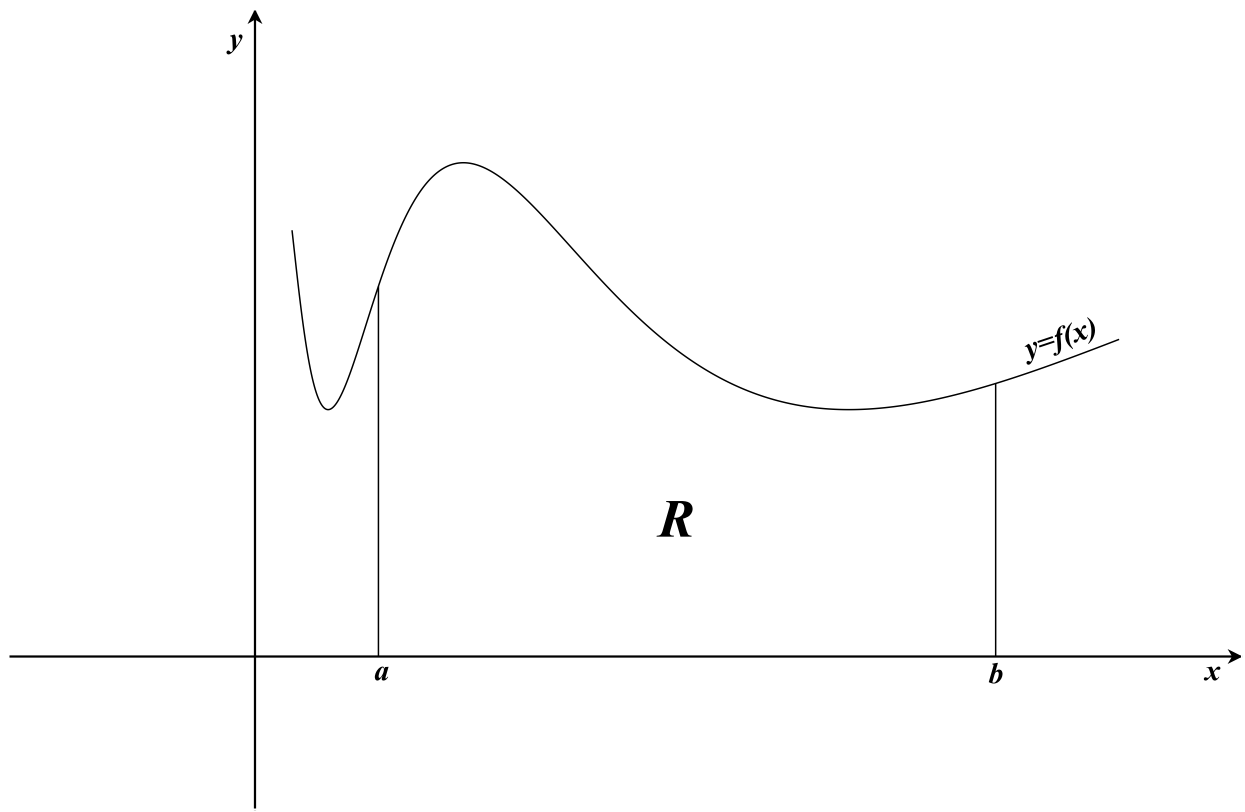
The Definite Integral

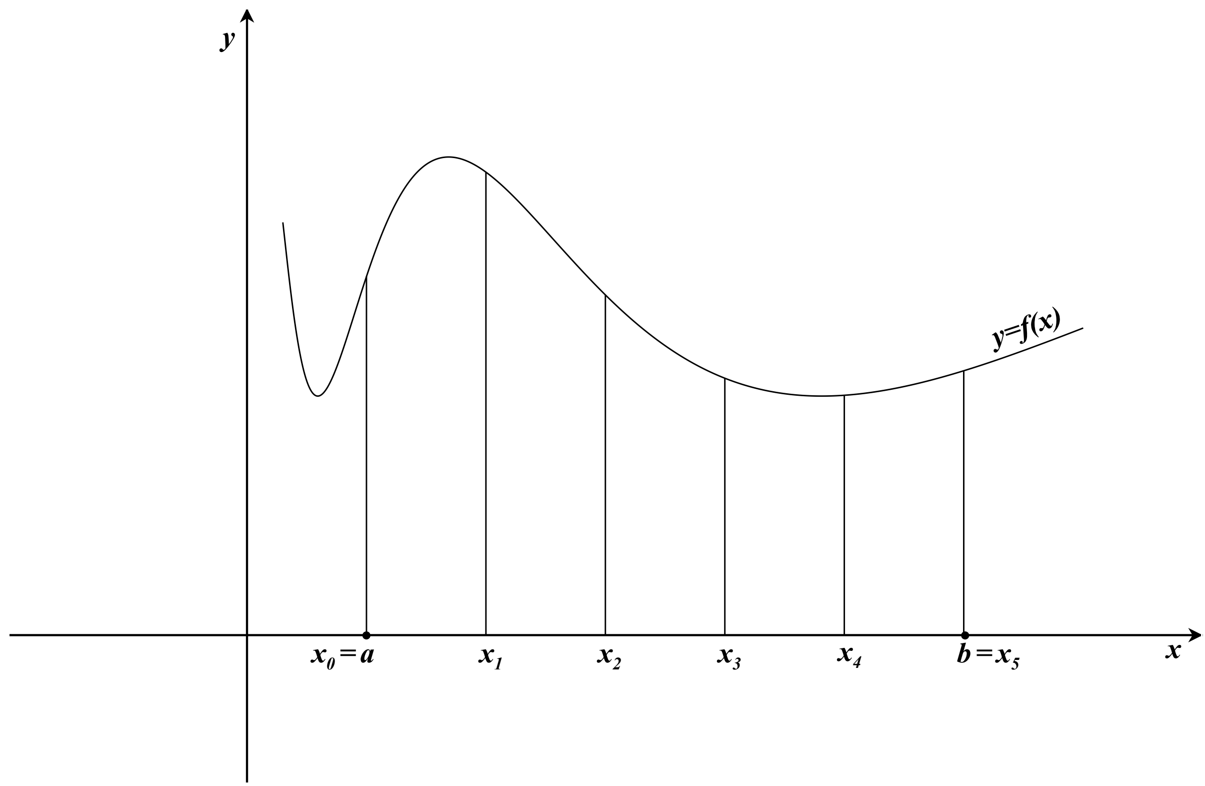
Motivational example:

Find the area of the region R, that is bounded by the lines $y = f(x)$, $x = a$, $x = b$ and the x-axis.



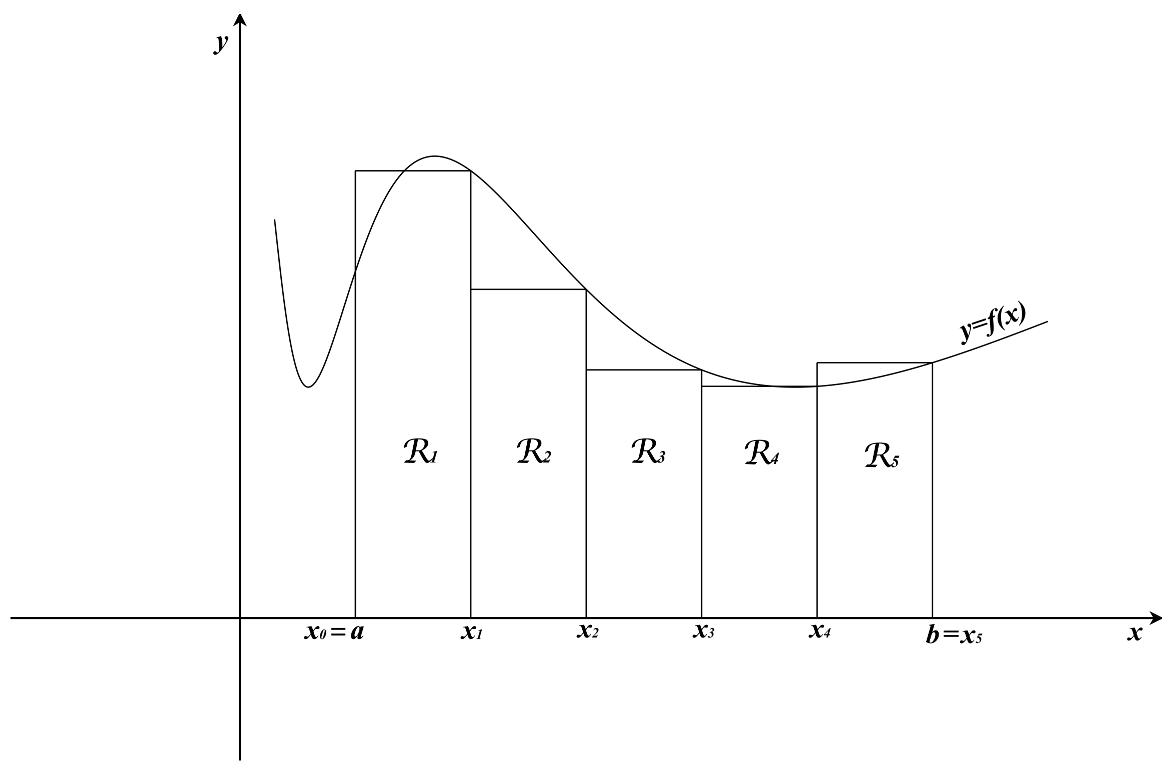
(✻) ***Problem*** — area is defined in terms of *rectangles*, and the region R is not rectangular.

(✻) ***Step 1, Approximation:*** Cover the region R with rectangles, and use the ***sum of the areas*** of these rectangles to ***approximate*** the area of R. To begin, we divide the region into *rectangular strips* ...



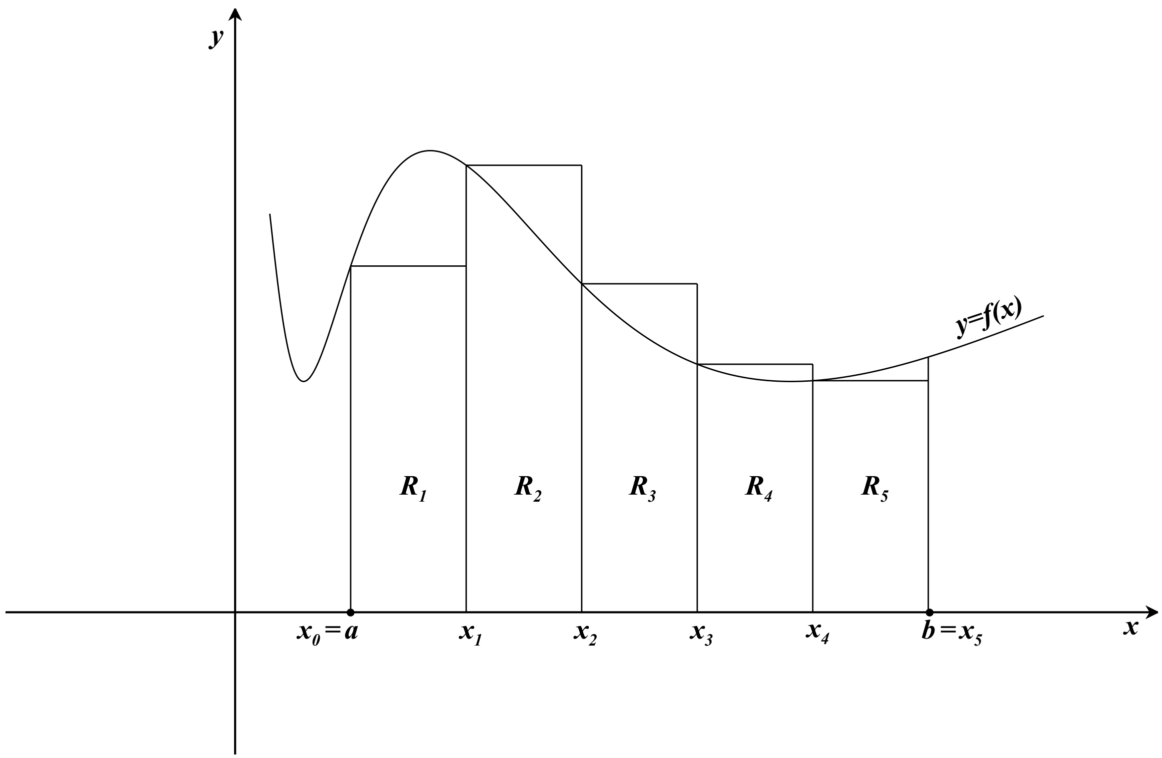
(✻) ... then approximate each strip by a rectangle, using a point on the graph of $y = f(x)$ to determine the height of each rectangle. This can be done in different ways. In figure **??**, the height of $\mathcal{R}_j$ is $f(x_j)$ (corresponding to the righthand endpoint of the base of $\mathcal{R}_j$).

Figure 1: Using righthand endpoints to determine heights.



In figure **??**, the height of R_j is $f(x_{j-1})$ (corresponding to the lefthand endpoint of the base of R_j).

Figure 2: Using lefthand endpoints to determine heights.



Next, using the second set of rectangles for example, we conclude that

$$\begin{aligned}\text{area}(R) &\approx \sum_{j=1}^{5} \text{area}(R_j) \\ &= \sum_{j=1}^{5} f(x_{j-1}) \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^{5} f(x_{j-1}) \cdot \Delta x_j,\end{aligned}$$

where, as usual, $\Delta x_j = (x_j - x_{j-1})$.

(✻) The sum in the third row above is called a ***lefthand sum***, because we used the lefthand endpoints to determine the heights of the rectangles.

(✻) If we use the rectangles in figure **??**, we obtain a ***righthand sum*** approximation

$$\text{area}(R) \approx \sum_{j=1}^{5} \text{area}(\mathcal{R}_j) = \sum_{j=1}^{5} f(x_j) \cdot \Delta x_j.$$

(✻) ***Step 2, Refine:*** To *improve* the approximate answer from Step 1, replace the original collection of rectangles by a bigger collection of rectangles, all of which are ***narrower*** than before, as in Figure **??**.

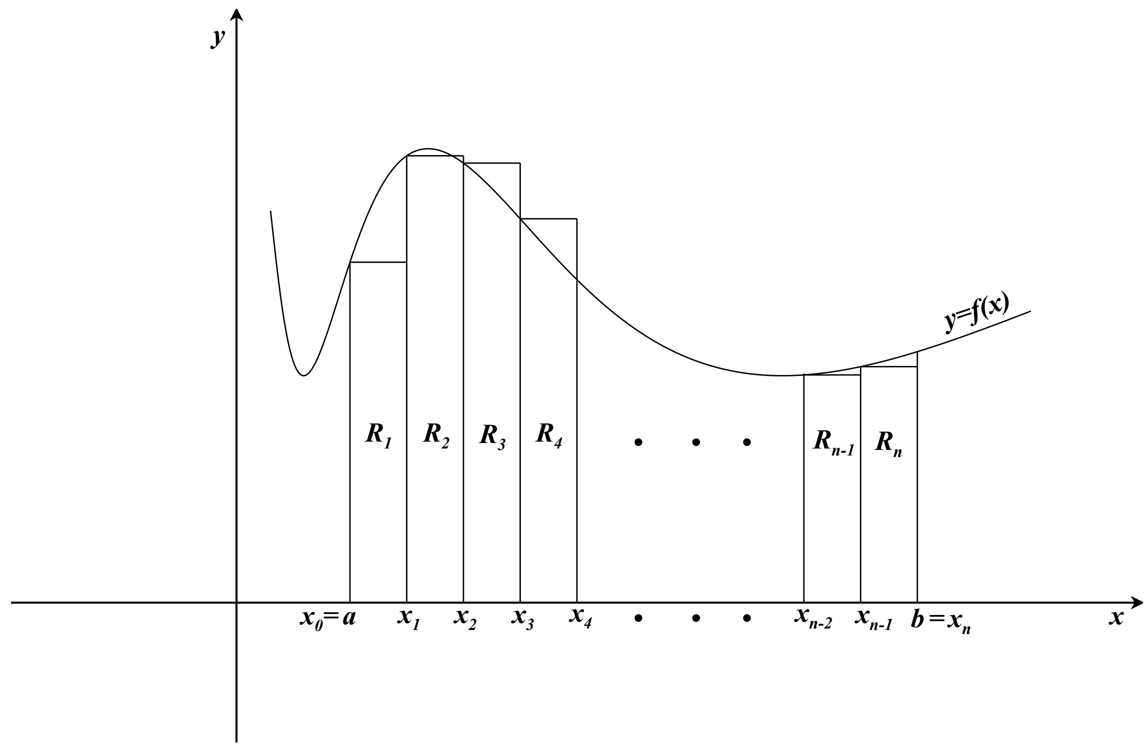


Figure 3: Refining the cover.

(✻) This is called '*refining the cover*'.

Using the rectangles in figure **??** gives the approximation

$$\boxed{\text{area}(R) \approx \sum_{j=1}^{n} \text{area}(R_j) = \sum_{j=1}^{n} f(x_{j-1}) \cdot \Delta x_j}$$

(another *lefthand sum*).

(✻) The refined approximation is typically more accurate than the previous approximation because the narrower rectangles cover the region more accurately.

What next?

Repeat... ***repeat...*** ***repeat...*** ***repeat...***

(✻) ***Step 3, take a limit:*** Continue to refine the collection of rectangles, making the corresponding approximations more and more accurate.

This leads to the conclusion that

$$\boxed{\text{area}(R) = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_{j-1})\cdot \Delta x_j\right),}$$

where

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$;

and it is understood that...

as we increase the number of rectangles, the widths of all the rectangles goes to 0:

$$\lim_{n\to\infty}\left[\max(\Delta x_j : 1 \le j \le n)\right] = 0$$

(✻) **Comment:** We don't have to use lefthand sums (or righthand sums).

Definition: The ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\,dx$$

and is defined by the limit

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\cdot \Delta x_j\right),} \tag{1}$$

where for each n:

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- x_j^* is *some point* in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \le x_j^* \le x_j$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$.
- $\lim_{n\to\infty} \big[\max(\Delta x_j : 1 \le j \le n)\big] = 0$.

(✻) The collection of subintervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ is called a *partition* of the original interval $[a, b]$. The length of longest subinterval in any given partition, $\max(\Delta x_j : 1 \le j \le n)$, is called the *diameter* of the partition.

Comments:

(✻) A definite integral $\int_a^b f(x)\,dx$ returns a ***numerical value***.

(✻) If the function $f(x)$ is *continuous* in the interval $[a, b]$, then the limit defining the definite integral *always exists*.

(✻) The value of the limit ***does not depend*** on how the *partitions* are chosen or how the points x_j^* are selected from each subinterval in each subinterval of the partition, as long as the *diameter* of the partition is approaching 0.

(✻) Computing area is an *application* of definite integrals — *not the way that they are defined.*

(✻) If $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a partition of $[a, b]$ and $x_{j-1} \leq x_j^* \leq x_j$ for each j, then the sum

$$\sum_{j=1}^{n} f(x_j^*) \cdot \Delta x_j$$

is called a *Riemann sum.*

(✻) **The most common choices:**

(✻) Divide the interval $[a, b]$ into n ***equal*** pieces. Then

$$\Delta x_j = \frac{b-a}{n}$$

and

$$x_j = a + j \cdot \Delta x_j = a + j \cdot \frac{b-a}{n}$$

for each j.

(✻) This also guarantees that $\Delta x_j \to 0$ as $n \to \infty$.

(✻) If $x_j^* = x_{j-1}$, then the resulting (Riemann) sum is a Left hand sum:

$$LHS(n) = \sum_{j=1}^{n} f(x_{j-1}) \Delta x.$$

(✻) If we set $x_j^* = x_j$, then the resulting (Riemann) sum is a Right hand sum:

$$RHS(n) = \sum_{j=1}^{n} f(x_j) \Delta x.$$

Example 1: Find the area of the region bounded by the curve $y = x^2$ and the lines $y = 0$, $x = 1$ and $x = 3$.

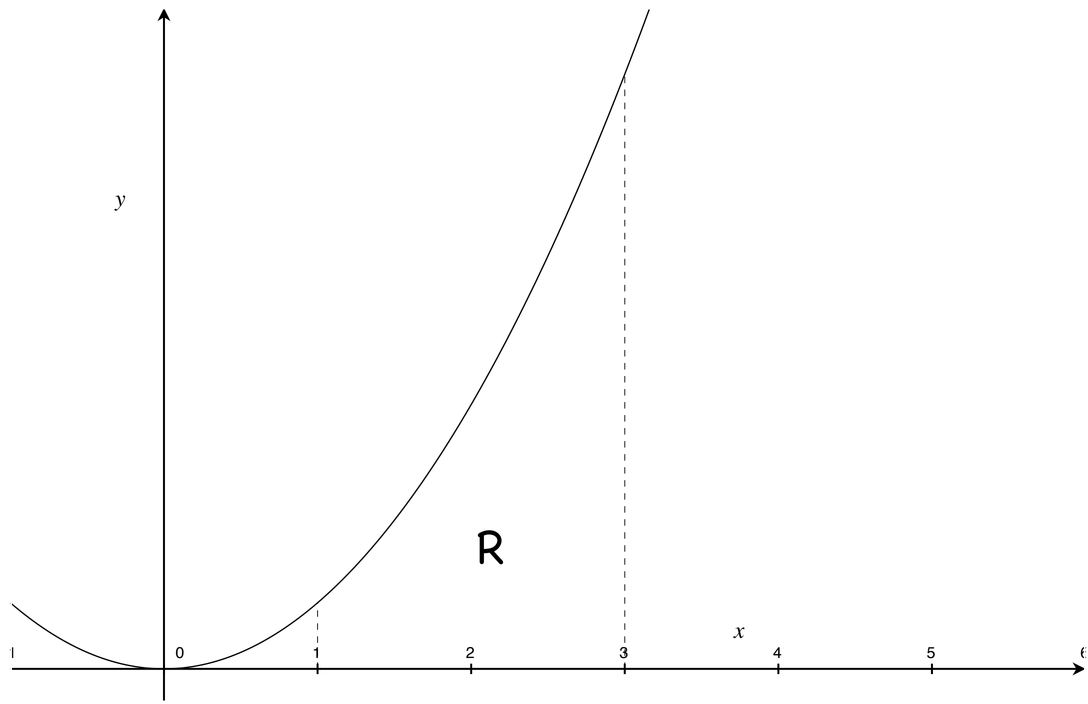


Figure 4: The region in Example 1.

(✻) Use *right hand sums* to calculate

$$\text{area}(\mathcal{R}) = \int_1^3 x^2\, dx.$$

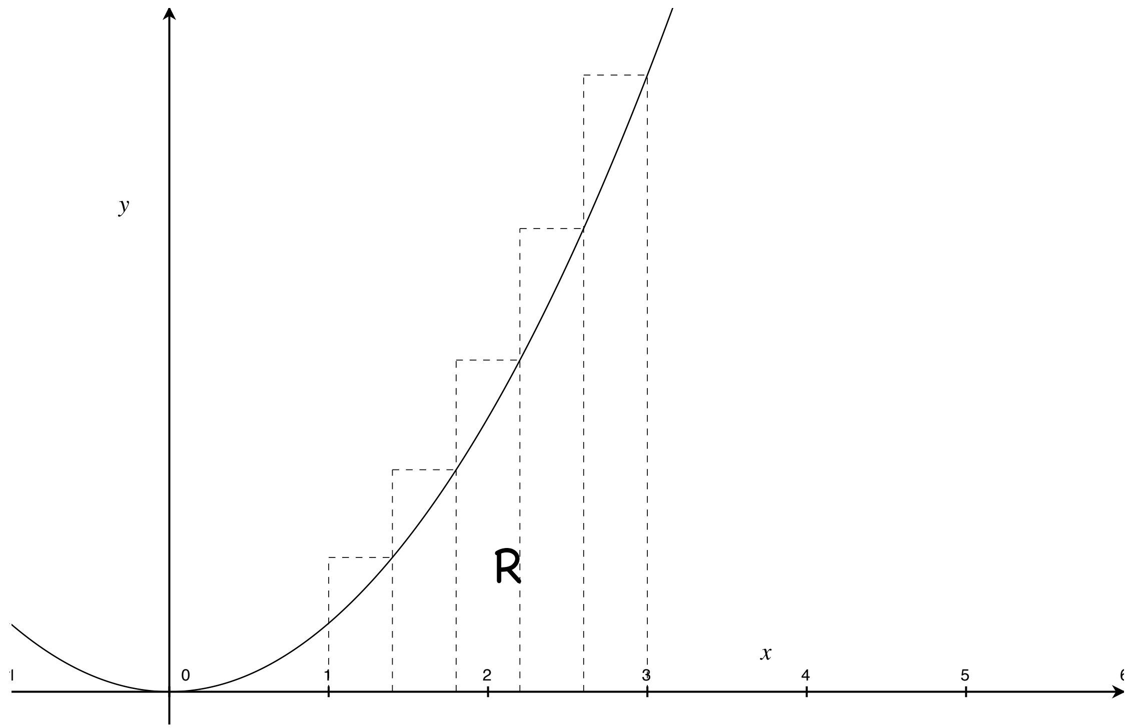


Figure 5: Right hand sum.

(✻) Divide the interval $[1, 3]$ into n equal pieces, then...

1. $\Delta x_j = \dfrac{3-1}{n} = \dfrac{2}{n}$, and

2. $x_j = 1 + j \cdot \dfrac{2}{n} = 1 + \dfrac{2j}{n}$.

(✻) Write down a right hand sum for this problem:

$$RHS(n) = \sum_{j=1}^{n} f(x_j)\Delta x_j = \sum_{j=1}^{n} x_j^2 \cdot \frac{2}{n} = \sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n}.$$

(✻) Simplify:

$$RHS(n) = \sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n} = \frac{2}{n} \sum_{j=1}^{n} \left(1 + \frac{4j}{n} + \frac{4j^2}{n^2}\right)$$

$$= \frac{2}{n} \left(\sum_{j=1}^{n} 1 + \frac{4}{n} \sum_{j=1}^{n} j + \frac{4}{n^2} \sum_{j=1}^{n} j^2 \right)$$

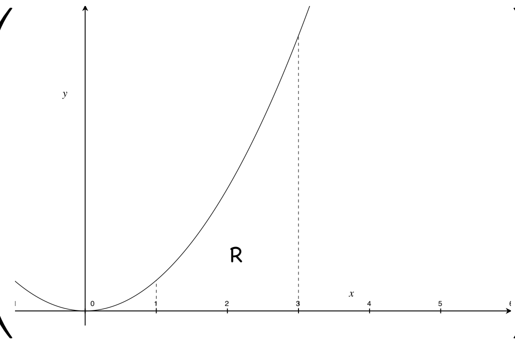
$$= \frac{2}{n} \cdot n + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= 2 + \frac{4n^2 + 4n}{n^2} + \frac{8n^3 + 12n^2 + 4n}{3n^3}$$

$$= 2 + 4 + \frac{8}{3} + \frac{4}{n} + \frac{4}{n} + \frac{4}{3n^2}$$

$$= \frac{26}{3} + \frac{8}{n} + \frac{4}{3n^2}$$

(✻) Take the limit:

area (

)

$$= \int_1^3 x^2\,dx$$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(1+\frac{2j}{n}\right)^2\cdot\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{26}{3}+\frac{8}{n}+\frac{4}{3n^2}\right)$$

$$= \frac{26}{3}$$