Differential calculus in several variables.

Example. A demand function

$$q = 5\sqrt{8y + 7p_s - 4p},$$

where

- q is the monthly demand for a firm's good, measured in 1000s of units.
- y is the average monthly income in the market for the firm's good, measured in 1000s of dollars.
- p_s is the average price of substitutes for a firm's good, measured in dollars.
- p is the price of the firm's good, measured in dollars.

(*) If the current prices are $p = \$10$ and $p_s = \$9$, and the average income is \$3300, then the demand will be

$$q\Big|_{\substack{y=3.3\\ ps=9\\ p=10}} = 5\sqrt{26.4 + 63 - 40} \approx 35.143 \implies \approx 35{,}143 \text{ units}$$

Question: What will happen to demand if the prices stay the same, but average income decreases to $y_1 = 3000$?

Qualitative answer: Demand will decrease.

Quantitative answer:

$$q\Big|_{\substack{y=3\\ps=9\\p=10}} = 5\sqrt{24+63-40} \approx 34.278 \implies \approx 34,278 \text{ units}$$

I.e., demand will decrease by about $35,143 - 34,278 = 865$ units.

(*) Key assumption: when calculating the effect of the change in income on the demand, we ***hold the other variables (the prices) fixed***.

(*) We used derivatives to analyze the behavior of functions of one variable, for example in optimization problems. We want to be able to do the same when there are more variables.

Definition: If $w = f(x, y, z)$, then the partial derivative of w with respect to x is defined by

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

Likewise, the partial derivatives of w with respect to y and z are

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

and

$$\frac{\partial w}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

Comments:

(i) We use the *'curly d'* symbol (∂) to distinguish partial derivatives from ordinary derivatives (of functions of one variable).

(ii) When differentiating with respect to one variable, the other variables are ***held fixed***.

(iii) The usual rules of differentiation still hold... Yay!

Example. Returning to the demand function

$$q = 5\sqrt{8y + 7p_s - 4p} = 5(8y + 7p_s - 4p)^{1/2},$$

The partial derivative of q with respect to income is

$$\frac{\partial q}{\partial y} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot 8 = \frac{20}{(8y + 7p_s - 4p)^{1/2}},$$

(*) I used the rule for powers and the chain rule, and most importantly, I treated p and p_s as constants.

I.e., when differentiating with respect to y, we can think of the demand function as being

$$q = 5(8y + C)^{1/2}$$

where $C = 7p_s - 4p$. Similarly, the other partial derivatives are

$$\frac{\partial q}{\partial p_s} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot 7 = \frac{35}{2(8y + 7p_s - 4p)^{1/2}}$$

and

$$\frac{\partial q}{\partial p} = 5 \cdot \frac{1}{2}(8y + 7p_s - 4p)^{-1/2} \cdot (-4) = -\frac{10}{(8y + 7p_s - 4p)^{1/2}}$$

Notation: For functions of one variable, we have two standard ways to denote the derivative. We can express the derivative of the function $y = f(x)$ as...

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx}.$$

(*) We have already extended the dy/dx notation to the case of functions of several variables with the $\partial w/\partial x$ notation.

(*) To extend the $f'(x)$ notation, we use something that indicates the variable with respect to which we are differentiating.

If $w = f(x, y, z)$, then we also write its partial derivatives as follows:

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} = f_x = w_x$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} = f_y = w_y$$

etc.

More examples...

1. $f(x,y,z) = 3x^2y^3z - 5y^2z^3$, find f_x, f_y and f_z.

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(3x^2y^3z - 5y^2z^3\right) = \frac{\partial}{\partial x}\left(3x^2y^3z\right) - \frac{\partial}{\partial x}\left(5y^2z^3\right) \\ &= 2x \cdot \left(3y^3z\right) - 0 = 6xy^3z \end{aligned}$$

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\left(3x^2y^3z - 5y^2z^3\right) = \frac{\partial}{\partial y}\left(3x^2y^3z\right) - \frac{\partial}{\partial y}\left(5y^2z^3\right) \\ &= 3y^2 \cdot \left(3x^2z\right) - 2y \cdot (5z^3) = 9x^2y^2z - 10yz^3 \end{aligned}$$

$$\begin{aligned} f_z &= \frac{\partial}{\partial z}\left(3x^2y^3z - 5y^2z^3\right) = \frac{\partial}{\partial z}\left(3x^2y^3z\right) - \frac{\partial}{\partial z}\left(5y^2z^3\right) \\ &= 3x^2y^3 - 15y^2z^2 \end{aligned}$$

2. $f(x,y) = \dfrac{x^2y - 3xy^2}{3y+1}$, find f_x and f_y.

$$f_x = \frac{\partial}{\partial x}\left(\frac{1}{3y+1}\cdot\left(yx^2 - 3y^2x\right)\right) = \frac{1}{3y+1}\cdot\frac{\partial}{\partial x}\left(yx^2 - 3y^2x\right)$$
$$= \frac{1}{3y+1}(y\cdot 2x - 3y^2) = \frac{2xy - 3y^2}{3y+1}$$

We don't need the quotient rule to find f_x because the denominator is not a function of the variable x. On the other hand, when we differentiate with respect to y, we do use the quotient rule:

$$f_y = \frac{\left[\frac{\partial}{\partial y}\left(x^2y - 3xy^2\right)\right]\cdot(3y+1) - \left(x^2y - 3xy^2\right)\cdot\left[\frac{\partial}{\partial y}(3y+1)\right]}{(3y+1)^2}$$
$$= \frac{(x^2 - 6xy)(3y+1) - (x^2y - 3xy^2)\cdot 3}{(3y+1)^2}$$
$$= \frac{3x^2y - 18xy^2 + x^2 - 6xy - 3x^2y + 9xy^2}{(3y+1)^2} = \frac{x^2 - 9xy^2 - 6xy}{(3y+1)^2}$$

3. $w = 3x^2 \ln(xy + 2z)$, find w_x, w_y and w_z.

$$\frac{\partial w}{\partial x} = 6x \ln(xy + 2z) + 3x^2 \cdot \frac{1}{xy + 2z} \cdot y = 6x \ln(xy + 2z) + \frac{3x^2 y}{xy + 2z}$$

We need to use the product rule (and the chain rule) to find $\partial w / \partial x$ because w is a product of two functions that depend on x.

On the other hand, we don't need to use the product rule to find $\partial w / \partial z$ and $\partial w / \partial y$, (but we still need the chain rule):

$$\frac{\partial w}{\partial y} = 3x^2 \cdot \frac{1}{xy + 2z} \cdot x = \frac{3x^3}{xy + 2z}$$

and

$$\frac{\partial w}{\partial z} = 3x^2 \cdot \frac{1}{xy + 2z} \cdot 2 = \frac{6x^2}{xy + 2z}.$$