Linear approximation (so far):

If $z = f(x, y)$ and $\Delta x \approx 0$ (with $\Delta y = 0$), then

$$\Delta z = f(x_0 + \Delta x, y_0) - f(x_0, y_0) \approx f_x(x_0, y_0) \cdot \Delta x.$$

Likewise, if $\Delta y \approx 0$ (with $\Delta x = 0$), then

$$\Delta z = f(x_0, y_0 + \Delta y) - f(x_0, y_0) \approx f_y(x_0, y_0) \cdot \Delta y.$$

Example: Suppose that a firm produces two competing goods, A and B, and that the firm's revenue function is given by $R(Q_A, Q_B)$, where Q_A and Q_B are the monthly demands for goods A and B.

If the demand for good A increases by one unit and the demand for good B remains fixed, then

$$\Delta R = R(Q_A + 1, Q_B) - R(Q_A, Q_B)$$

is called the *marginal revenue of good A*.

On the other hand,

$$\Delta R \approx \frac{\partial R}{\partial Q_A} \cdot \Delta Q_A = \frac{\partial R}{\partial Q_A} \cdot 1 = \frac{\partial R}{\partial Q_A}.$$

Just as in the one variable case, we call the partial derivative $\partial R/\partial Q_A$ a marginal revenue function. More specifically we say that $\partial R/\partial Q_A$ is the marginal revenue of good A.

For the same reason, we call $\partial R/\partial Q_B$ the marginal revenue of good B.

Suppose, for example that

$$R(Q_A, Q_B) = 640Q_A + 460Q_B - 4Q_A^2 - 6Q_AQ_B - 2Q_B^2,$$

where Q_A and Q_B are both measured in 1000s of units.

In this example, the marginal revenue functions are

$$\frac{\partial R}{\partial Q_A} = 640 - 8Q_A - 6Q_B \text{ and } \frac{\partial R}{\partial Q_B} = 460 - 6Q_A - 4Q_B.$$

If the demand for A increases from 5000 units to 5400 units, while the demand for B remains fixed at 4000 units, then

$$\Delta R \approx \frac{\partial R}{\partial Q_A}\bigg|_{\substack{Q_A=5 \\ Q_B=4}} \cdot \Delta Q_A = 576 \cdot (0.4) = 230.4$$

Similarly, if the demand for A remains fixed at 5000 units, while the demand for B increases from 4000 units to 4500 units, then

$$\Delta R \approx \frac{\partial R}{\partial Q_B}\bigg|_{\substack{Q_A=5 \\ Q_B=4}} \cdot \Delta Q_B = 414 \cdot (0.5) = 207.$$

Question: Since we are assuming that A and B are *competing* goods, what typically happens to the demand for B if the demand for A increases?

If the demand for A increases, then it is common to see a decrease in demand for B and vice versa.

Is it possible to extend linear approximation to the case where both A and B are changing?

In other words:

If $z = f(x,y)$, then how can we approximate

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

if both $\Delta x \approx 0$ and $\Delta y \approx 0$, but neither are $= 0$?

Answer: (more general linear approximation)

$$\Delta z \approx f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \cdot \Delta y.$$

Explanation: To approximate Δz, break it into two components:

$$\begin{aligned}
\Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\
&= \overbrace{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}^{\Delta_1} + \overbrace{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)}^{\Delta_2} \\
&\approx f_x(x_0, y_0) \cdot \Delta x + f_y(x_0 + \Delta x, y_0) \cdot \Delta y \\
&\approx f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \cdot \Delta y
\end{aligned}$$

because $f_y(x_0 + \Delta x, y_0) \approx f_y(x_0, y_0)$ if $\Delta x \approx 0$ (and f_y is continuous).

Returning to the revenue example...

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Then

$$\Delta R \approx \left.\frac{\partial R}{\partial Q_A}\right|_{\substack{Q_A=5\\Q_B=4}} \cdot \Delta Q_A + \left.\frac{\partial R}{\partial Q_B}\right|_{\substack{Q_A=5\\Q_B=4}} \cdot \Delta Q_B$$

$$= 576 \cdot (0.4) + 414 \cdot (-0.3) = 106.2$$

Example: The monthly household demand for widgets is given by

$$q(y,p) = 5\sqrt{y^2 - 2p}$$

where q = demand; y = monthly income (\$1000s); p = price of a widget.

- If monthly disposable income is \$4000 and the price of a widget is \$6, then the household's monthly demand for widgets is

$$q(4,6) = 5\sqrt{16-12} = 5\sqrt{4} = 10.$$

- If household income increases to \$4300 and the price decreases to \$5.75, by approximately how much will demand increase?
- First: $q_y = 5y(y^2-2p)^{-1/2}$ and $q_p = -5(y^2-2p)^{-1/2}$.
- Second: $q_y(4,6) = 10$ and $q_p(4,6) = -5/2$
- Third: $\Delta y = \frac{300}{1000} = 0.3$ and $\Delta p = -0.25$, so...

  $\Delta q \approx q_y(4,6)\cdot\Delta y + q_p(4,6)\cdot\Delta p = 10\cdot 0.3 + (-5/2)(-0.25) = 3.625$

This general form of linear approximation applies to any number of variables, e.g., if $w = f(x, y, z)$ and Δx, Δy and Δz are small, then

$$\begin{aligned} \Delta w &= f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \\ &\approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z. \end{aligned}$$

Writing $x_0 + \Delta x = x$, $y_0 + \Delta y = y$ and $z_0 + \Delta z = z$, so that

$$\Delta x = x - x_0,\ \Delta y = y - y_0 \text{ and } \ \Delta z = z - z_0,$$

we can rewrite the linear approximation formula above as

$$\begin{aligned} f(x, y, z) - f(x_0, y_0, z_0) \approx f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ + f_z(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

or

$$\begin{aligned} f(x, y, z) &\approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ &\qquad + f_z(x_0, y_0, z_0)(z - z_0) \\ &= T_1(x, y, z) \end{aligned}$$

The linear function

$$T_1(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) \\ + f_z(x_0, y_0, z_0)(z - z_0)$$

is the ***linear Taylor polynomial*** for $w = f(x, y, z)$ centered at (x_0, y_0, z_0), and linear approximation (in three variables) can be written as

$$f(x, y, z) \approx T_1(x, y, z),$$

assuming that $x \approx x_0$, $y \approx y_0$ and $z \approx z_0$.

We will use this approximation to justify the definition of *critical points*, in the context of optimization.

To generalize the *second derivative test* to two variables and beyond, we will need the ***quadratic*** Taylor polynomial in two (or more) variables.

For this we need higher order partial derivatives.

The (*first order*) partial derivatives of a function $z = f(x, y)$ are

$$f_x = z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad f_y = z_y = \frac{\partial z}{\partial y}.$$

The *second order* partial derivatives of a function $z = f(x, y)$ are (not surprisingly) the partial derivatives of its (first order) partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = (z_x)_x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{yx} = (z_y)_x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{xy} = (z_x)_y$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy} = (z_y)_y$$

Example. If $z = 4x^3 + 3x^2y - 2xy^2 + y^3$, then its (first order) partial derivatives are

$$z_x = 12x^2 + 6xy - 2y^2 \quad \text{and} \quad z_y = 3x^2 - 4xy + 3y^2$$

and its second order partial derivatives are

$$z_{xx} = 24x+6y, \quad z_{yx} = 6x-4y, \quad z_{xy} = 6x-4y \quad \text{and} \quad z_{yy} = -4x+6y.$$

Observation: In this example, $z_{xy} = z_{yx}$.

*Coincidence? ...**No.***

Fact:

> *Second and higher order partial derivatives do not depend on the order with respect to which a function is differentiated, only on **the number of times** the function is differentiated with respect to each variable.*

The *third order* partial derivatives of a function of two or more variables are the partial derivatives of its second order partial derivatives.

Notation: For $z = f(x, y)$

$$z_{xxx} = \frac{\partial^3 z}{\partial x^3}, \qquad \overbrace{z_{xyx} = z_{xxy}}^{\text{aforementioned fact}} = \frac{\partial^3 z}{\partial x^2 \partial y}, \quad \textbf{\textit{etc.}}$$

Example. (continued) For $z = 4x^3 + 3x^2y - 2xy^2 + y^3$ we already know that

$$z_{xx} = 24x + 6y, \quad z_{yx} = 6x - 4y = z_{xy} \quad \text{and} \quad z_{yy} = -4x + 6y.$$

so

$$z_{xxx} = 24,\ z_{yxx} = 6,\ z_{xyx} = 6,\ z_{yyx} = -4,$$

$$z_{xxy} = 6,\ z_{yxy} = -4,\ z_{xyy} = -4 \text{ and } z_{yyy} = 6$$

Note that

$$z_{yxx} = z_{xyx} = z_{xxy} = 6 \quad \text{and} \quad z_{yyx} = z_{yxy} = z_{xyy} = -4,$$

as the 'Fact' predicted.