Example 1. Compute $\displaystyle\int \frac{x^2+3x+1}{2x+1}\,dx$

Approach 1. Substitution: $u = 2x+1 \implies du = 2dx \implies dx = \frac{1}{2}du$ and $x = \frac{1}{2}(u-1)$:

$$\begin{aligned}
\int \frac{x^2+3x+1}{2x+1}\,dx &= \frac{1}{2}\int \frac{\left(\frac{1}{2}(u-1)\right)^2 + \frac{3}{2}(u-1)+1}{u}\,du \\
&= \frac{1}{2}\int \frac{\frac{1}{4}(u^2-2u+1)+\frac{3}{2}u-\frac{1}{2}}{u}\,du \\
&= \frac{1}{2}\int \frac{\frac{1}{4}u^2+u-\frac{1}{4}}{u}\,du \\
&= \frac{1}{2}\int \frac{1}{4}u + 1 - \frac{1/4}{u}\,du \\
&= \frac{1}{16}u^2 + \frac{1}{2}u - \frac{1}{8}\ln|u| + C \\
&= \frac{1}{16}(2x+1)^2 + \frac{1}{2}(2x+1) - \frac{1}{8}\ln|2x+1| + C
\end{aligned}$$

Approach 2. Long division of polynomials:

$$
\begin{array}{r@{\,}l}
 & \quad\ \ \tfrac{1}{2}x + \tfrac{5}{4} \\ \cline{2-2}
2x+1\big) & x^2 + 3x + 1 \\
 & -x^2 - \tfrac{1}{2}x \\ \cline{2-2}
 & \qquad \tfrac{5}{2}x + 1 \\
 & \qquad -\tfrac{5}{2}x - \tfrac{5}{4} \\ \cline{2-2}
 & \qquad\qquad -\tfrac{1}{4}
\end{array}
$$

which means that $\dfrac{x^2+3x+1}{2x+1} = \dfrac{1}{2}x + \dfrac{5}{4} - \dfrac{1/4}{2x+1}$, and therefore

$$
\begin{aligned}
\int \frac{x^2+3x+1}{2x+1}\,dx &= \int \frac{1}{2}x + \frac{5}{4} - \frac{1/4}{2x+1}\,dx \\
&= \frac{1}{4}x^2 + \frac{5}{4}x - \frac{1}{4}\int \frac{1}{2x+1}\,dx \\
&= \frac{1}{4}x^2 + \frac{5}{4}x - \frac{1}{8}\int \frac{1}{u}\,du \quad \text{(substituting } u = 2x+1 \text{ again)} \\
&= \frac{1}{4}x^2 + \frac{5}{4}x - \frac{1}{8}\ln|2x+1| + C
\end{aligned}
$$

Observations:

First, note that

$$\begin{aligned}\frac{1}{16}(2x+1)^2+\frac{1}{2}(2x+1) &= \frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{16}+x+\frac{1}{2}\\ &= \frac{1}{4}x^2+\frac{5}{4}x+\frac{9}{16}\end{aligned}$$

so

$$\frac{1}{16}(2x+1)^2+\frac{1}{2}(2x+1)-\frac{1}{8}\ln|2x+1|+C = \frac{1}{4}x^2+\frac{5}{4}x-\frac{1}{8}\ln|2x+1|+C$$

since $9/16$ can be absorbed by C. I.e., the answers we got using the two approaches are the same.

Second, some algebra is unavoidable whichever approach you choose, so pick the flavor you like the best, and practice it. Or practice both, for good measure.

Example 2. The marginal propensity to consume for a small nation is given by

$$\frac{dC}{dY} = \frac{8Y + 15}{9Y + 1},$$

where consumption C and income Y are both measured in billions of dollars. Find the total change in the nation's consumption and saving if income increases from \$10 billion to \$15 billion.

Observations:

1. We can't determine the consumption function completely (we don't have data to solve for the constant of integration), but we do have enough information to find the *change* in consumption because if $C = f(Y) + K$, then

$$\Delta C = C(Y_2) - C(Y_1) = f(Y_2) + K - (f(Y_2) + K) = f(Y_2) - f(Y_1).$$

2. If S is national saving, then $C + S = Y$, so $S = Y - C$ and

$$\begin{aligned}\Delta S = S(Y_2) - S(Y_1) &= (Y_2 - C(Y_2)) - (Y_1 - C(Y_1)) \\ &= (Y_2 - Y_1) - (C(Y_2) - C(Y_1)) = \Delta Y - \Delta C.\end{aligned}$$

I'll use the substitution $u = 9Y + 1 \implies dY = \frac{1}{9}\,du$ and $Y = \frac{1}{9}(u-1)$ to integrate:

$$\begin{aligned}
\int \frac{8Y+15}{9Y+1}\,dY &= \frac{1}{9}\int \frac{\frac{8}{9}(u-1)+15}{u}\,du \\
&= \frac{1}{9}\int \frac{8}{9} + \frac{127/9}{u}\,du = \frac{1}{81}u + \frac{127}{81}\ln|u| + K \\
&= \frac{8}{81}(9Y+1) + \frac{127}{81}\ln|9Y+1| + K \\
&= \frac{8}{9}Y + \frac{127}{81}\ln|9Y+1| + K
\end{aligned}$$

This means that $C(Y) = f(Y) + K$, where $f(Y) = \frac{8}{9}Y + \frac{127}{81}\ln|9Y+1| + K$ and therefore

$$\begin{aligned}
\Delta C &= C(15) - C(10) = f(15) - f(10) \\
&= \left(\frac{120}{9} + \frac{127}{81}\ln 136\right) - \left(\frac{80}{9} + \frac{127}{81}\ln 91\right) \\
&\approx 5.074
\end{aligned}$$

and $\Delta S = \Delta Y - \Delta C \approx 5 - 5.074 = -0.074$. (*What does this mean?*)

Exponential functions.

Example 3. Compute $\int e^{ax}\,dx$, where $a \neq 0$.

The substitution $u = ax \implies du = a\,dx \implies dx = \frac{1}{a}\,du$ works here:

$$\int e^{ax}\,dx = \frac{1}{a}\int e^{u}\,du = \frac{1}{a}e^{u} + C = \frac{1}{a}e^{ax} + C.$$

Example 4. Compute $\int b^{x}\,dx$, where $b > 0$ and $b \neq 1$.

In this case, we begin with the observation

$$\ln b^{x} = x\ln b \implies b^{x} = e^{\ln b^{x}} = e^{x\ln b} = e^{(\ln b)x}.$$

Therefore

$$\int b^{x}\,dx = \int e^{(\ln b)x}\,dx = \frac{1}{\ln b}e^{(\ln b)x} + C = \frac{1}{\ln b}b^{x} + C,$$

using the formula from the previous example with $a = \ln b$.