

## Techniques of integration: Substitution

**Observation:** If  $u = g(x)$ , then  $du = g'(x) dx$ .

We can use this observation to simplify integrals that have the form

$$\int f(g(x)) g'(x) dx,$$

by *substituting*  $u = g(x)$ , which entails  $du = g'(x) dx$ :

$$\int \overbrace{f(g(x))}^u \overbrace{g'(x) dx}^{du} = \int f(u) du$$

**Example 1:** The substitution  $u = x^2 + 3$ ,  $du = 2x dx$  gives

$$\begin{aligned} \int 2x \sqrt{x^2 + 3} dx &= \int \overbrace{(x^2 + 3)}^u \overbrace{2x dx}^{du} \\ &= \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C \\ &= \frac{2}{3} (x^2 + 3)^{3/2} + C \end{aligned}$$

(\*) The key is *recognizing* that a substitution is possible.

(\*) What we are looking for in the integrand is a *product*, with one factor being a *composite function*, e.g.,  $f(g(x))$ , and the other factor being the *derivative of the argument* of the first factor, i.e.,  $g'(x)$ .

**Example 2:** Compute

$$\int \frac{2x}{x^2 + 4} dx$$

The integrand looks like a quotient, but really it's a product, and one of the factors is composite:

$$\int \frac{2x}{x^2 + 4} dx = \int (x^2 + 4)^{-1} 2x dx$$

Substituting  $u = x^2 + 4$  and  $du = 2x dx$  makes the integral easy to compute:

$$\int \overbrace{(x^2 + 4)^{-1}}^u \overbrace{2x dx}^{du} = \int u^{-1} du = \ln |u| + C = \ln(x^2 + 4) + C.$$

**Observation:** We can multiply or divide any equation by a *nonzero* constant. This makes substitution more flexible.

**Example 3:** Compute

$$\int 5x^2 \sqrt[4]{2x^3 + 1} dx$$

The natural choice for a substitution is  $u = 2x^3 + 1$ , which means that  $du = 6x^2 dx$ . Dividing this relation by 6 shows that

$$x^2 dx = \frac{1}{6} du \implies 5x^2 dx = \frac{5}{6} du,$$

so the substitution  $u = 2x^3 + 1$  will work here:

$$\begin{aligned} \int 5x^2 \sqrt[4]{2x^3 + 1} dx &= \int \overbrace{(2x^3 + 1)^{1/4}}^u \overbrace{5x^2 dx}^{\frac{5}{6} du} \\ &= \frac{5}{6} \int u^{1/4} du = \frac{2}{3} u^{5/4} + C \\ &= \frac{2}{3} (2x^3 + 1)^{5/4} + C \end{aligned}$$

**Observation:** If  $u = ax + b$ , then  $du = a dx$  so that  $dx = \frac{1}{a} du$ . This means that we can always make the following simplification

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

by substituting  $u = ax + b$ .

**Example 4:** Compute

$$\int \sqrt[3]{5x + 2} dx = \int (5x + 2)^{1/3} dx.$$

The substitution  $u = 5x + 2$ ,  $du = 5 dx \implies dx = \frac{1}{5} du$  will work here:

$$\int \overbrace{(5x + 2)}^u \overbrace{dx}^{\frac{1}{5} du} = \int u^{1/3} \cdot \frac{1}{5} du = \frac{3}{20} u^{4/3} + C = \frac{3}{20} (5x + 2)^{4/3} + C$$

**Example 5:** Find the demand equation for a firm whose marginal revenue function is

$$\frac{dr}{dq} = 10 \sqrt[3]{1000 - 0.5q}.$$

**Observation:** Since  $r = pq$ , by definition, we can find the demand equation for this firm by

- (i) Finding the revenue function  $r(q)$ , and
- (ii) Writing  $p = r(q)/q$ .

**Step 1.** Integrate, by substituting  $u = 1000 - 0.5q$ ,  $du = -0.5 dq \implies dq = -2 du$ :

$$\begin{aligned} \int 10 \sqrt[3]{1000 - 0.5q} dq &= 10 \int \overbrace{(1000 - 0.5q)}^u \overset{-2 du}{dq}^{1/3} \\ &= -20 \int u^{1/3} du \\ &= -15u^{4/3} + C = -15(1000 - 0.5q)^{4/3} + C \end{aligned}$$

**Conclusion 1:**  $r = C - 15(1000 - 0.5q)^{4/3}$

**Step 2:** Solve for  $C$ :

$$0 = r(0) = C - 15(1000)^{4/3} \implies C = 150000.$$

**Conclusion:**  $r = 150000 - 15(1000 - 0.5q)^{4/3}$ .

**Step 3:** Write down the demand equation:

$$p = \frac{r}{q} = \frac{15(10000 - (1000 - 0.5q)^{4/3})}{q}$$

**Observation:** Linear equations are easy to solve. For example, if  $u = ax + b$ , then

$$x = \frac{1}{a}(u - b).$$

**Example 6:** Compute

$$\int \frac{2x}{3x+4} dx.$$

The substitution

$$u = 3x + 4, \quad du = 3 dx \implies dx = \frac{1}{3} du$$

can also be used to express  $2x$  in terms of  $u$ :

$$x = \frac{1}{3}(u - 4) \implies 2x = \frac{2}{3}(u - 4),$$

so

$$\begin{aligned} \int \underbrace{\frac{2x}{3x+4}}_u \underbrace{dx}_{\frac{1}{3} du} &= \int \frac{\frac{2}{3}(u-4)}{u} \frac{1}{3} du = \frac{2}{9} \int 1 - \frac{4}{u} du \\ &= \frac{2}{9}u - \frac{8}{9} \ln |u| + C = \frac{2}{9}(3x+4) - \frac{8}{9} \ln |3x+4| + C \\ &= \frac{2}{3}x - \frac{8}{9} \ln |3x+4| + C \end{aligned}$$