The Indefinite Integral

- If $F'(x) = f(x)$, then $F(x)$ is called an ***antiderivative*** of $f(x)$.

- If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C.

- On the other hand, if $F'(x) = G'(x) = f(x)$, then

$$\frac{d}{dx}(G(x) - F(x)) = f(x) - f(x) = 0,$$

so $G(x) - F(x) = C$ (a constant). I.e., $G(x) = F(x) + C$.

- The ***indefinite integral*** of $f(x)$ is the set of *all* antiderivatives of $f(x)$, and denoted by $\int f(x)\,dx$. If we know that $F(x)$ is an antiderivative of $f(x)$, then we write

$$\int f(x)\,dx = F(x) + C$$

Basic rules of integration:

1. $\displaystyle\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

2. $\displaystyle\int \alpha f(x)\,dx = \alpha \int f(x)\,dx,$ for any constant $\alpha \neq 0$.

3. $\displaystyle\int x^k\,dx = \frac{x^{k+1}}{k+1} + C,$ for any constant power $k \neq -1$.

3.1 $\displaystyle\int 1\,dx = \int x^0\,dx = x + C.$

3.2 $\displaystyle\int \alpha\,dx = \alpha \int x^0\,dx = \alpha x + \alpha C = \alpha x + C.$

4. $\displaystyle\int x^{-1}\,dx = \ln|x| + C$

5. $\displaystyle\int e^x\,dx = e^x + C$

Examples:

1. $$\int x^2 + 3x + 4\,dx = \int x^2\,dx + \int 3x\,dx + \int 4\,dx$$
$$= \int x^2\,dx + 3\int x\,dx + \int 4\,dx$$
$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x + C$$

2. $$\int 4\sqrt[5]{x} - \frac{3}{x^2}\,dx = \int 4x^{1/5}\,dx - \int 3x^{-2}\,dx = 4\int x^{1/5}\,dx - 3\int x^{-2}\,dx$$
$$= 4\cdot\frac{x^{6/5}}{6/5} - 3\cdot\frac{x^{-1}}{-1} + C = \frac{10}{3}x^{6/5} + \frac{3}{x} + C$$

3. $$\int \frac{2x^2 + 3x - 5}{4x}\,dx = \int \frac{2x^2}{4x} + \frac{3x}{4x} - \frac{5}{4x}\,dx = \int \frac{1}{2}x + \frac{3}{4} - \frac{5}{4}x^{-1}\,dx$$
$$= \frac{1}{2}\int x\,dx + \int \frac{3}{4}\,dx - \frac{5}{4}\int x^{-1}\,dx$$
$$= \frac{1}{4}x^2 + \frac{3}{4}x - \frac{5}{4}\ln|x| + C$$