The Definite Integral: Definition

The ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\,dx$$

and is defined by the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\cdot\Delta x_j\right),$$

where for each n:

- $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$;
- x_j^* is *some point* in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \le x_j^* \le x_j$;
- $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$.
- $\lim_{n\to\infty}\big[\max(\Delta x_j : 1 \le j \le n)\big] = 0$.

Comment: Computing definite integrals this way is usually computationally intensive and frequently very difficult (if not impossible).

Happily, there is

The Fundamental Theorem of Calculus:

If $F'(x) = f(x)$, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Observations:

- The value of $\int_a^b f(x)\,dx$ does not depend on which antiderivative of $f(x)$ we use, because if $G(x) = F(x) + C$, then

$$G(b) - G(a) = (F(b) + \cancel{C}) - (F(a) + \cancel{C}) = F(b) - F(a).$$

- This connection between the definite integral and anti-differentiation is why we use the same integral sign for both definite and indefinite integrals.

The *Fundamental Theorem of Calculus* (FTC) can also be expressed as follows.

If $f(x)$ is a given (continuous) function, then

$$F(x) = \int_a^x f(t)\,dt$$

is a function of x, where the lower limit of integration, a, is held constant. In this case,

- $F'(x) = f(x)$

  This statement requires a bit of work to prove, but is interesting for those that want to understand a bit more of what is going on behind the scene — see section 14.7 in the textbook.

- $F(a) = 0$, and
- $\displaystyle\int_a^b f(x)\,dx = F(b) = F(b) - F(a)$

(*) This is how one proves (if one is so inclined) that every continuous function has antiderivatives.

Example 1. Use the FTC to calculate $\int_1^3 x^2\,dx$.

We know that $\int x^2\,dx = \frac{x^3}{3} + C$, so

$$\int_1^3 x^2\,dx = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27-1}{3} = \frac{26}{3}.$$

Notation: We denote the difference $F(b) - F(a)$ by

$$F(b) - F(a) = \left. F(x)\right|_a^b .$$

This makes applying the FTC a little more smooth notationally speaking.

In the previous example we can write:

$$\int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}.$$

Example 2. Compute $\displaystyle\int_0^2 2x^3 - 3x^2 + 4x - 5\,dx$.

Using the FTC we have

$$\begin{aligned}\int_0^2 2x^3 - 3x^2 + 4x - 5\,dx &= \left(\frac{1}{2}x^4 - x^3 + 2x^2 - 5x\right)\Big|_0^2\\ &= \left(\frac{16}{2} - 8 + 8 - 10\right) - (0 - 0 + 0 - 0)\\ &= -2\end{aligned}$$

Example 3. Compute $\displaystyle\int_1^4 \frac{x-2}{\sqrt{x}}\,dx$.

Using the FTC we have

$$\begin{aligned}\int_1^4 \frac{x-2}{\sqrt{x}}\,dx &= \int_1^4 x^{1/2} - 2x^{-1/2}\,dx\\ &= \left(\frac{2}{3}x^{3/2} - 4x^{1/2}\right)\Big|_1^4\\ &= \left(\frac{16}{3} - 8\right) - \left(\frac{2}{3} - 4\right) = \frac{2}{3}\end{aligned}$$

Justification of the FTC:

(*) *Linear approximation:* if $x_j - x_{j-1} = \Delta x_j$ is small, then

$$F'(x_{j-1})\Delta x_j \approx F(x_j) - F(x_{j-1}).$$

(*) *Telescoping sums:* If $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, then

$$\begin{aligned}\sum_{j=1}^{n}(F(x_j) - F(x_{j-1})) &= (\cancel{F(x_1)} - F(x_0)) + (\cancel{F(x_2)} - \cancel{F(x_1)}) + (\cancel{F(x_3)} - \cancel{F(x_2)}) \\ &\quad + \cdots + (\cancel{F(x_{n-1})} - \cancel{F(x_{n-2})}) + (F(x_n) - \cancel{F(x_{n-1})}) \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a)\end{aligned}$$

(*) If n is very large and all the Δx_j s are small, then

$$\sum_{j=1}^{n} F'(x_{j-1})\Delta x_j \approx \sum_{j=1}^{n}(F(x_j) - F(x_{j-1})) = F(b) - F(a)$$

and as $n \to \infty$, the sum on the left approaches the constant value on the right. In other words,

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n} F'(x_{j-1})\Delta x_j\right) = F(b) - F(a)$$

(*) On the other hand, if $F'(x) = f(x)$, then from the *Definition* of the definite integral (using left-hand sums) we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_{j-1})\Delta x_j\right)\\ &= \lim_{n\to\infty}\left(\sum_{j=1}^{n} F'(x_{j-1})\Delta x_j\right) = F(b) - F(a).\end{aligned}$$

Comment: This is the outline of a proof, but some important details are missing. For example, while it is true that $F'(x_{j-1})\Delta x_j \approx F(x_j) - F(x_{j-1})$ when Δx_j is sufficiently small, there is still a small error. This means that the approximation

$$\sum_{j=1}^{n} F'(x_{j-1})\Delta x_j \approx \sum_{j=1}^{n}(F(x_j) - F(x_{j-1}))$$

entails a sum of n small errors. The explanation of why the sum of these n errors is (very) small when n is large is a missing detail.

Substitution in a definite integral.

Example 4. Compute $\displaystyle\int_0^1 \frac{5}{\sqrt{3x+1}}\,dx$

Once again, using the FTC and the substitution

$$u = 3x+1 \implies dx = \frac{1}{3}du,$$

we have

$$\int_0^1 \frac{5}{\sqrt{3x+1}}\,dx = \frac{5}{3}\int_?^? u^{-1/2}\,du = \frac{10}{3}u^{1/2}\Big|_?^? = ?$$

(*) When making a substitution in a definite integral, ***the limits of integration change with the substitution***. E.g., in this problem, if $u = 3x+1$, then $x = 0 \implies u = 1$ and $x = 1 \implies u = 4$, so

$$\int_0^1 \frac{5}{\sqrt{3x+1}}\,dx = \frac{5}{3}\int_1^4 u^{-1/2}\,du = \frac{10}{3}u^{1/2}\Big|_1^4 = \frac{20}{3} - \frac{10}{3} = \frac{10}{3}$$

Example 5. Compute $\int_0^{20} e^{-0.04t}\,dt$.

Use the substitution $u = -0.04t$, so $dt = -25\,du$. This also entails changing the limits of integration:

$$t = 0 \implies u = 0 \text{ and } t = 20 \implies u = -0.8.$$

Thus

$$\begin{aligned}\int_0^{20} e^{-0.04t}\,dt &= -25\int_0^{-0.8} e^u\,du \\ &= -25e^u\Big|_0^{-0.8} \\ &= -25e^{-0.8} - (-25e^0) \approx 13.767\end{aligned}$$

(*) The lower limit of integration does not have to be smaller than the upper limit of integration in a definite integral (though it usually is). The FTC works just fine in both cases, as in the example above.

Properties of definite integrals:

These properties are easy to justify using the FTC, but they can all be justified using the definition as well. The first two are direct analogs of the same properties for indefinite integrals, while the last two do not have indefinite integral counterparts.

1. $\displaystyle\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2. $\displaystyle\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$