Differential equations.

A first order ordinary differential equation is an equation involving a function, its derivative and the free variable it depends on:

$$\Phi\left(y, \frac{dy}{dx}, x\right) = 0,$$

where it is assumed that $y = y(x)$.

The equation is ***separable*** *if it can be algebraically manipulated to be put in the form:*

$$h(y)\,dy = g(x)\,dx,$$

i.e., if the variables (y and x in this case) can be ***separated****.*

Example 1. The differential equation

$$\frac{y'}{x} - 3y = 0 \tag{1}$$

is separable because we can manipulate it as follows:

$$\frac{dy/dx}{x} - 3y = 0 \implies \frac{1}{x} \cdot \frac{dy}{dx} = 3y \implies \frac{1}{y}\,dy = 3x\,dx$$

A ***solution*** of a differential equation is a function $y = f(x)$ that satisfies the equation.

Example 1. (cont.) The function $y = e^{3x^2/2}$ is a solution of Equation (1) because $y' = 3xe^{3x^2/2}$ (chain rule), so

$$\frac{y'}{x} = \frac{3xe^{3x^2/2}}{x} = 3e^{3x^2/2} = 3y.$$

Observation: If A is any constant, then the function $y_A = Ae^{3x^2/2}$ is also a solution of Equation (1), because

$$y'_A = 3Axe^{3x^2/2} \implies \frac{y'_A}{x} = \frac{3Axe^{3x^2/2}}{x} = 3Ae^{3x^2/2} = 3y'_A.$$

Fact: If a differential equation has a solution, then it generally has infinitely many.

Why?

Because solving differential equations involves (indefinite) integration which entails an unknown constant of integration, leading to infinitely many different solutions.

Solving separable equations. To solve an equation of the form $h(y)\,dy = g(x)\,dx$...

(i) *Integrate both sides (if possible):*

$$h(y)\,dy = g(x)\,dx \implies \int h(y)\,dy = \int g(x)\,dx \implies H(y) = G(x) + C.$$

The resulting equation, $H(y) = G(x)+C$ is called an ***implicit solution***.

(ii) Solve the implicit equation for y (if possible):

$$y = H^{-1}\big(G(x) + C\big)\,.$$

This is called an ***explicit solution*** (H^{-1} is the *inverse function* of H).

Example 1. (cont.) Integrating both sides of $\frac{1}{y}\,dy = 3x\,dx$ gives

$$\int \frac{1}{y}\,dy = \int 3x\,dx \implies \ln|y| = \frac{3x^2}{2} + C \quad \text{(Implicit solution).}$$

To find the ***explicit*** solution, *exponentiate* both sides of the implicit solution:

$$\implies |y| = e^{\ln|y|} = e^{(3x^2/2)+C} = e^C \cdot e^{3x^2/2} \implies y = Ae^{3x^2/2}, \quad (A = \pm e^C)$$

Why differential equations?

Mathematical models are based on theoretical assumptions about the variables in question. These assumptions are framed in terms of how the model ***changes****, and change is described by the derivatives of the functions involved.*

In physics, for example, Newton's law of cooling (and heating) states that the temperature T of a body immersed in a medium of constant temperature τ changes at a rate that is proportional to the difference $T - \tau$. In this case, the temperature of the body is a function of time t, and Newton's law can be written as

$$\frac{dT}{dt} = k(T - \tau),$$

where k is the constant of proportionality.

Theoretical assumptions in economics can also lead to differential equations...

Example 2. The *labor-elasticity of output* for a certain industry is assumed to be constant. Find the production function $q = f(l)$.

The elasticity of output q with respect to labor input l is defined as

$$\eta_{q/l} = \frac{dq}{dl} \cdot \frac{l}{q}.$$

The assumption of *constant elasticity* leads to the (separable) equation

$$\eta_{q/l} = \eta_0 \implies \frac{dq}{dl} \cdot \frac{l}{q} = \eta_0 \implies \frac{dq}{q} = \eta_0 \frac{dl}{l},$$

where η_0 is the (unknown) constant elasticity.

Integrating both sides of the right hand equation gives the implicit solution

$$\int \frac{dq}{q} = \eta_0 \int \frac{dl}{l} \implies \ln q = \eta_0 \ln l + C.$$

Exponentiation gives the explicit solution

$$e^{\ln q} = e^{\eta_0 \ln l + C} \implies q = e^C \cdot e^{\ln(l^{\eta_0})} \implies q = A l^{\eta_0},$$

where (as before) $A = e^C$.

To obtain a more precise mathematical model for the phenomenon being studied, scientists combine theoretical assumptions with ***data***. The data is used to find specific values for the (as-yet-unknown) parameters that appear in the (general) solutions of the differential equations.

Example 1. (cont.) We found that the (general) solution of the differential equation (1) is

$$y = Ae^{3x^2/2}.$$

To pick out a specific solution, we need to specify a value of the function at some point, called an *initial value*. For example, if require that our solution $y = f(x)$ of (1) also satisfy $f(0) = 2$, then we find only one value of A that works:

$$2 = f(0) = Ae^{3\cdot 0^2/2} = Ae^0 = A.$$

I.e., $y = 2e^{3x^2/2}$ is the *unique* solution of the ***initial value problem***

$$\frac{y'}{x} - 3y = 0; \quad y(0) = 2.$$

Observation: This situation is analogous to what we saw in the beginning of the quarter with indefinite integrals.

If $f(x)$ is continuous, then there are infinitely many solutions to the (differential) equation

$$y' = f(x)$$

given by the indefinite integral

$$y = \int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any antiderivative of $f(x)$.

But there is only one solution to the initial value problem

$$y' = f(x); \quad y(x_0) = y_0,$$

namely

$$y = F(x) + (y_0 - F(x_0)).$$

The amount of data that we require to find a specific solution depends on the number of unspecified parameters that appear in the general (implicit or explicit) solution of the differential equation.

Example 2. (cont.) We saw that if labor-elasticity of output ($\eta_{q/l}$) is constant, then the production function must have the form

$$q = Al^{\eta_0},$$

where η_0 is the value of the constant elasticity. If η_0 is known, then we just need one data point to find A, but if η_0 is unknown, we will need two data points...

Suppose that the production function we seek also satisfies $q(20) = 200$ and $q(30) = 250$. This data leads to a pair of equations for the unknown parameters A and η_0, which we solve as follows:

$$\left.\begin{array}{rcl} A \cdot 20^{\eta_0} & = & 200 \\ A \cdot 30^{\eta_0} & = & 250 \end{array}\right\} \Longrightarrow \frac{\not{A} 30^{\eta_0}}{\not{A} 20^{\eta_0}} = \frac{250}{200} \Longrightarrow \left(\frac{3}{2}\right)^{\eta_0} = \frac{5}{4}.$$

Taking logarithms of both sides of the right-most equation gives

$$\eta_0 \ln(3/2) = \ln(5/4) \Longrightarrow \eta_0 = \frac{\ln(5/4)}{\ln(3/2)} \quad (\approx 0.55034),$$

and

$$A = 200 \cdot 20^{-\eta_0} \approx 38.461.$$

I.e.,

$$q \approx 38.461 \cdot l^{0.55034}$$

is the production function we seek.

Summary: To solve the (separable) initial value(s) problem

$$h(y)y' - g(x) = 0; \quad y(x_0) = y_0,\, y(x_1) = y_1, \ldots, y(x_n) = y_n$$

(i) Separate:

$$h(y)y' - g(x) = 0 \implies h(y)\frac{dy}{dx} = g(x) \implies h(y)\,dy = g(x)\,dx.$$

(ii) Integrate (if possible):

$$h(y)\,dy = g(x)\,dx \implies \int h(y)\,dy = \int g(x)\,dx \implies H(y) = G(x) + C.$$

(iii) Solve for y (if possible):

$$H(y) = G(x) + C \implies y = H^{-1}\big(G(x) + C\big)\,.$$

(iv) Use data to solve for C (and any other unspecified parameters in H and G). The number of data points should generally match the number of unknown constants.